Game Theory

Lecture 4: Nash equilibrium – a definition

Frieder Neunhoeffer

Lisbon School of Economics & Management
Universidade de Lisboa

UNIVERSIDADE DE LISBOA

Nash equilibrium

A *equilibrium strategy* for a player means that she expects each player chooses a *best reply* in relation to the *expectations* or *beliefs* that each player has about the strategies selected by her opponents.

A **Nash equilibrium** is a strategy profile x, such that the strategy of each player x_i

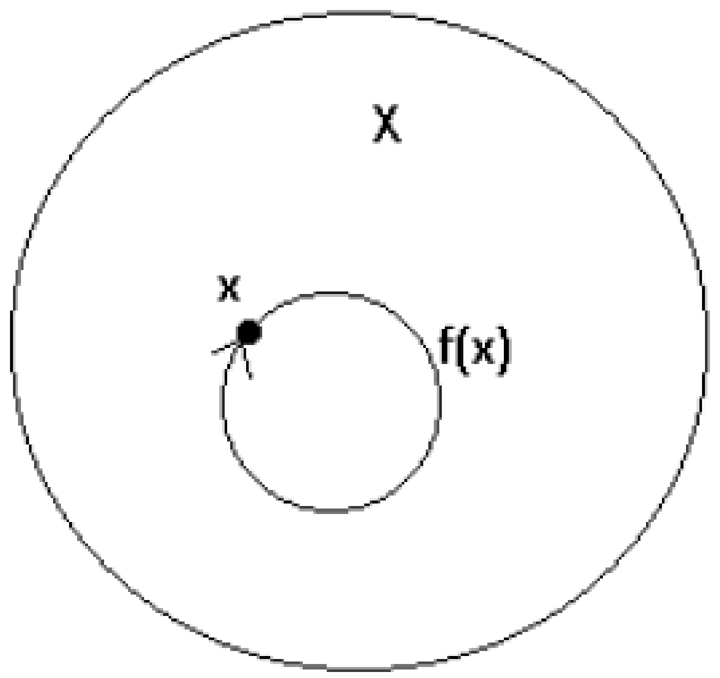
1. does not only correspond to the *subjective beliefs* that player i holds about the moves by the other players,
2. but also it is optimal under the assumption that the other players select their equilibrium strategies x_{-i}.

Formal definition of Nash equilibrium

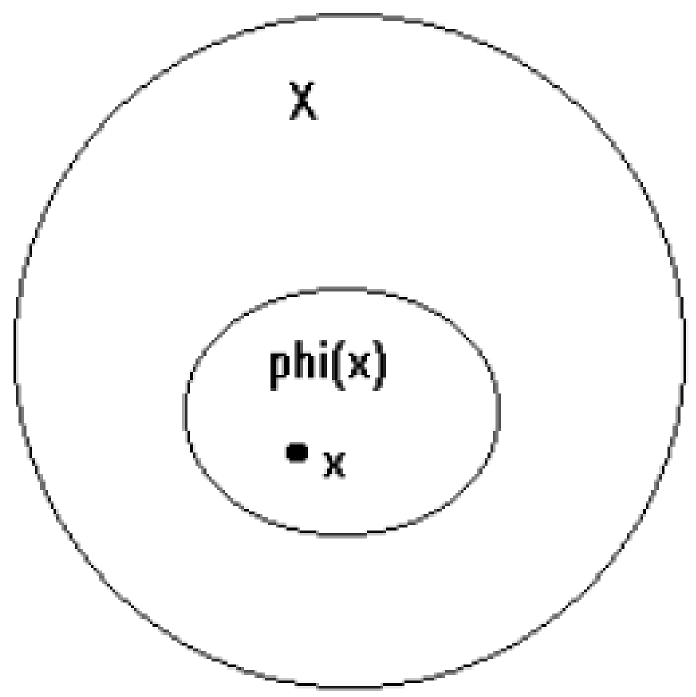
We need first to define the notion of "fixed point" of a function or of a correspondence.

Definition of fixed point: A *fixed point* of a function $f(x)$ is a variable x such that it is coincident with the image: $x = f(x)$. Function $f(.)$ "transports“ the variable x into itself, so that it is kept "fixed". The fixed point of a correspondence $\phi(x)$ is a variable x such that the image set contains the variable: $x \in \phi(x)$.

Definition of Nash equilibrium: A profile of mixed strategies x is a Nash equilibrium if it is a best reply to itself, i.e., if it is a fixed point of the correspondence of best replies $\tilde{\beta}$. $x \in \Theta$ is a Nash equilibrium if $x \in \tilde{\beta}(x)$.



Fixed point of a function



Fixed point of a correspondence

Practical meaning

Consider the profile of mixed strategies $x = (x_1, x_2, \ldots, x_n)$. Now, calculate:

(1) All possible values of x_1^*, i.e., the best reply of player 1 to $(x_2, \ldots, x_n)$.

(2) All possible values of x_2^*, i.e., the best reply of player 2 to $(x_1, x_3, \ldots, x_n)$.

...

(n) All possible values of x_n^*, i.e., the best reply of player n to $(x_1, \ldots, x_{n-1})$.

Now, consider all the profiles $x^* = (x_1^*, x_2^*, \ldots, x_n^*)$ that can be formed with the best replies. If the initial profile x is identical to one of the best reply profiles x^*, then x is a Nash equilibrium.

We are defining the Nash equilibrium in a general way, supposing that players can select mixed strategies. But we can use the concept supposing that each player is constrained to use only *pure strategies*.

Nash equilibrium in pure and mixed strategies

Assuming that each player can use mixed strategies, what is the relationship between Nash equilibrium in mixed and pure strategies?

If $x \in \Theta$ is a Nash equilibrium in mixed strategies, then any pure strategy in the support of the mixed strategy x_i, is a best reply to x. In formal terms, $s_i \in C(x_i) \Rightarrow s_i \in \beta_i(x)$.

What is the meaning of this? Consider a two-person game with a Nash equilibrium in *completely mixed* strategies. Then, the equilibrium strategy of each player is called an *equalizing strategy*, since it determines that the pure strategies of the opponent have the same expected payoff.

Although the concept of Nash equilibrium is intuitive, it is reasonable to ask the question:

Why do players aim for the Nash equilibrium?

Why do players coordinate on the Nash equilibrium?

The answer to this question is not unique (see VIVES, 2001):

1. The Nash equilibrium is a "necessary condition" (i.e., a *consistency requirement*) of any pattern of behavior where players are rational. If a result is not a Nash equilibrium, at least one player does not behave rationally because she does not maximize her payoff.

2. If players can communicate before they play, the Nash equilibrium is a "self-enforceable agreement", in the sense that each one gains by choosing the strategy that is prescribed by the Nash equilibrium, assuming that all the other players behave in the same way.

3. With no pre-play communication, assuming that the Nash equilibrium is unique, the players converge to it on account of the rationality assumption of *common knowledge* of the rules of the game by all players. This means that,

Why do players coordinate on the Nash equilibrium?

a) All players know the rules of the game (in two-person finite games, the payoff matrix).

b) Each player knows that each other player knows the rules of the game.

c) Each player knows that each other player knows that each other player knows the rules of the game. And so on .....

d) Under these assumptions, each player detects rationally the Nash equilibrium, by considering not only her point of view, but also the perspectives of the other players, and converges to this equilibrium point: it is the so called **"rational and idealized" interpretation** of the Nash equilibrium (NE).

e) It should be remarked that this interpretation requires that the NE is **unique**.

4. Even without the assumptions of **rationality** and **common knowledge of the game rules**, the NE can emerge out of the **repeated interaction** of players that are matched repeatedly and randomly: this is the so called **"mass action“ interpretation** of the NE.

Finding the Nash equilibrium in pure strategies

Finite games

The procedure to find a Nash equilibrium is straightforward. Consider a two-person finite game. Then, the normal form is:

	L	C	R
T	0,4	4,0	5,3
M	4,0	0,4	5,3
B	3,5	3,5	6,6

The Nash equilibrium in pure strategies can always be found by checking each outcome, represented by an element of the matrix. Any element such that no player gains by deviating is a **NE**.

A more efficient way consists in determining the best reply correspondences in pure strategies of the two players (i.e., $\beta_1(s)$ and $\beta_2(s)$).

Finding the Nash equilibrium in pure strategies

Finite games

$\beta_1(s)$ can be found by underlining the best replies of player 1 to each strategy ("matrix column") selected by player 2. The result is given by the matrix:

$$\begin{array}{cccc} & L & C & R \\ T & 0{,}4 & \underline{4}, 0 & 5{,}3 \\ M & \underline{4}, 0 & 0{,}4 & 5{,}3 \\ B & 3{,}5 & 3{,}5 & \underline{6}, 6 \end{array}$$

Likewise, $\beta_2(s)$ can be found by underlining the best replies of player 2 to each strategy ("matrix row") selected by player 1:

$$\begin{array}{cccc} & L & C & R \\ T & 0, \underline{4} & 4{,}0 & 5{,}3 \\ M & 4{,}0 & 0, \underline{4} & 5{,}3 \\ B & 3{,}5 & 3{,}5 & 6, \underline{6} \end{array}$$

Finding the Nash equilibrium in pure strategies

Finite games

$\beta(s) = \beta_1(s) \times \beta_2(s)$ can be determined by consolidating the two matrices with pure best replies to obtain:

	L	C	R
T	$0, \underline{4}$	$\underline{4}, 0$	$5,3$
M	$\underline{4}, 0$	$0, \underline{4}$	$5,3$
B	$3,5$	$3,5$	$\underline{6}, \underline{6}$

It is clear that the unique fixed point $s^* = \beta(s^*)$ is given by the profile of pure strategies $s^* = (B, R)$ with payoffs (6,6).

Continuous games

The same process of defining a fixed point can be used to locate the NE in a game with *continuous pure strategies*. Consider the so called *Cournot oligopoly,* where each player is a firm that supplies a homogeneous good. The normal form is given by:

$$I = \{1, 2, \ldots, n\} \qquad \text{set of firms}$$

$$S_i = [0, \infty) \qquad \text{set of pure strategies of firm } i$$

$$\pi_i(s) = s_i F\left(\sum_{i=1}^{n} s_i\right) - C_i(s_i) \qquad \text{payoff function of firm } i$$

The pure strategy of firm $i, s_i \in S_i$, is its output quantity of the good. We label s the profile of strategies of all firms $s = (s_1, s_2, \ldots, s_i)$.

The payoff function π_i of firm i is its profit function.

$F(\sum_{i=1}^{n} s_i)$ is the inverse demand function addressed to the industry.

$C_i(s_i)$ is the total cost function of firm i.

Reaction function

If we assume that the profit function of each firm i has a unique maximum in relation to its own output, the correspondence of best replies in pure strategies $\beta_i(s)$ can be found by solving the equation:

$$\frac{\partial \pi_i}{\partial q_i}(s_i, s_{-i}) = 0 \text{ to get}$$

$$s_i = R_i(s_{-i}) \text{ the } \textbf{reaction function} \text{ of firm } i$$

The set of the reaction functions of all firms,

$$\begin{aligned} s_1 &= R_1(s_2, \dots, s_n) \\ s_2 &= R_2(s_1, s_3 \dots, s_n) \\ &\dots \\ s_n &= R_n(s_1, \dots, s_{n-1}) \end{aligned}$$

forms the best reply correspondence $\beta(s)$, whose fixed point can be found by solving the equation system.

Experiment treatment 1: Monopoly

Each participant has the role of a monopoly seller in a market with a constant cost of $1 per unit, $c(q) = q$, and a simulated linear demand curve with a random shock, $E(\varepsilon) = 0$,

$$p = 13 - q + \varepsilon$$
$$\pi(q) = pq - c(q)$$
$$\pi(q) = (13 - q)q - q = 12q - q^2$$
$$\frac{\partial \pi}{\partial q} = 12 - 2q$$
$$\max_q \pi \rightarrow q = 6$$

Experiment treatment 2: Duopoly

What happens if a second firm enters the market?

Both firms have constant marginal costs of $1. Each selects an output quantity q_1, q_2.

$$p = 13 - (q_1 + q_2) + \varepsilon$$

$$\pi_1(q_1) = \big(13 - (q_1 + q_2)\big)q_1 - q_1 = 12q_1 - (q_1)^2 - q_1 q_2$$

$$\frac{\partial \pi_1}{\partial q_1} = 12 - 2q_1 - q_2$$

$$\frac{\partial \pi_1}{\partial q_1} = 0$$

$$q_1 = \frac{12 - q_2}{2} = 6 - \frac{q_2}{2}$$

$$\text{in equilibrium it must be} \rightarrow q_1 = q_2$$

$$\rightarrow q_1 = 4$$

$$q_2 = R_2(q_1 = 4) = 4$$

Cournot Duopoly: profit matrix and best responses

What happens if a second firm enters the market?

		Firm 2 (entrant)							
	Chosen quantity by each firm	*0*	*1*	*2*	*3*	*4*	*5*	*6*	…
Firm 1 (incumbent)	*0*								…
	1								…
	2		18, 9	16, 16	14, 21	12, 24	10, 25	8, 24	…
	3		24, 8	21, 14	18, 18	15, 20	12, 20	9, 18	…
	4		28, 7	24, 12	20, 15	16, 16	12, 15	8, 12	…
	5		30, 6	25, 10	20, 12	15, 12	10, 10	5, 6	…
	6		30, 5	24, 8	18, 9	12, 8	6, 5	0, 0	…
	…	…	…	…	…	…	…	…	

Cournot Duopoly: profit matrix and best responses

What happens if a second firm enters the market?

		Firm 2 (entrant)							
	Chosen quantity by each firm	*0*	*1*	*2*	*3*	*4*	*5*	*6*	…
Firm 1 (incumbent)	*0*	0, 0	0, 11	0, 20	0, 27	0, 32	0, 35	0, 36	…
	1	11, 0	10, 10	9, 18	8, 24	7, 28	6, 30	5, 35	…
	2	20, 0	18, 9	16, 16	14, 21	12, 24	10, 25	8, 24	…
	3	27, 0	24, 8	21, 14	18, 18	15, 20	12, 20	9, 18	…
	4	32, 0	28, 7	24, 12	20, 15	16, 16	12, 15	8, 12	…
	5	35, 0	30, 6	25, 10	20, 12	15, 12	10, 10	5, 6	…
	6	36, 0	30, 5	24, 8	18, 9	12, 8	6, 5	0, 0	…
	…	…	…	…	…	…	…	…	

Cournot Duopoly: profit matrix and best responses

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		Firm 2 (entrant)							
	Chosen quantity by each firm	*0*	*1*	*2*	*3*	*4*	*5*	*6*	…
Firm 1 (incumbent)	*0*	0, 0	0, 11	0, 20	0, 27	0, 32	0, 35	0, 36	…
	1	11, 0	10, 10	9, 18	8, 24	7, 28	6, 30	5, 35	…
	2	20, 0	18, 9	16, 16	14, 21	12, 24	10, 25	8, 24	…
	3	27, 0	24, 8	21, 14	18, 18	15, 20	12, 20	9, 18	…
	4	32, 0	28, 7	24, 12	20, 15	16, 16	12, 15	8, 12	…
	5	35, 0	30, 6	25, 10	20, 12	15, 12	10, 10	5, 6	…
	6	36, 0	30, 5	24, 8	18, 9	12, 8	6, 5	0, 0	…
	…	…	…	…	…	…	…	…	

Finding the Nash equilibrium in mixed strategies

In the *Matching Pennies* game, each of two children shows one face of a coin simultaneously. If both show the same face (i.e., two "Heads" or two "Tails"), child 2 wins. Otherwise, child 1 wins. The payoff matrix is:

		Child 2	
		Heads	*Tails*
Child 1	*Heads*	$-1,1$	$1,-1$
	Tails	$1,-1$	$-1,1$

There is no Nash equilibrium in pure strategies. In an equilibrium of completely mixed strategies, the strategy of each player should equalize the expected payoffs of the opponent's pure strategies.

Expected payoffs

If we assume that child 2 plays his pure strategy "Heads“ with probability q, the expected payoffs of Child 1’s pure strategies are,

$$E(\pi_1(heads)) = q(-1) + (1-q)1 = 1 - 2q$$
$$E(\pi_1(tails)) = q(1) + (1-q)(-1) = 2q - 1$$

Vice versa, if we assume that Child 1 uses "Heads“ with probability r, the expected payoffs of Child 2’s pure strategies are,

$$E(\pi_2(heads)) = r(1) + (1-r)(-1) = 2r - 1$$
$$E(\pi_2(tails)) = r(-1) + (1-r)1 = 1 - 2r$$

Best reply correspondences

Hence, the best reply correspondence of Child 1 is given by,

$$r = \beta_1(r,q) = \begin{cases} 1 & if\ 1-2q > 2q-1 \Leftrightarrow q < \frac{1}{2} \\ [0,1] & if\ 1-2q = 2q-1 \Leftrightarrow q = \frac{1}{2} \\ 0 & if\ 1-2q < 2q-1 \Leftrightarrow q > \frac{1}{2} \end{cases}$$

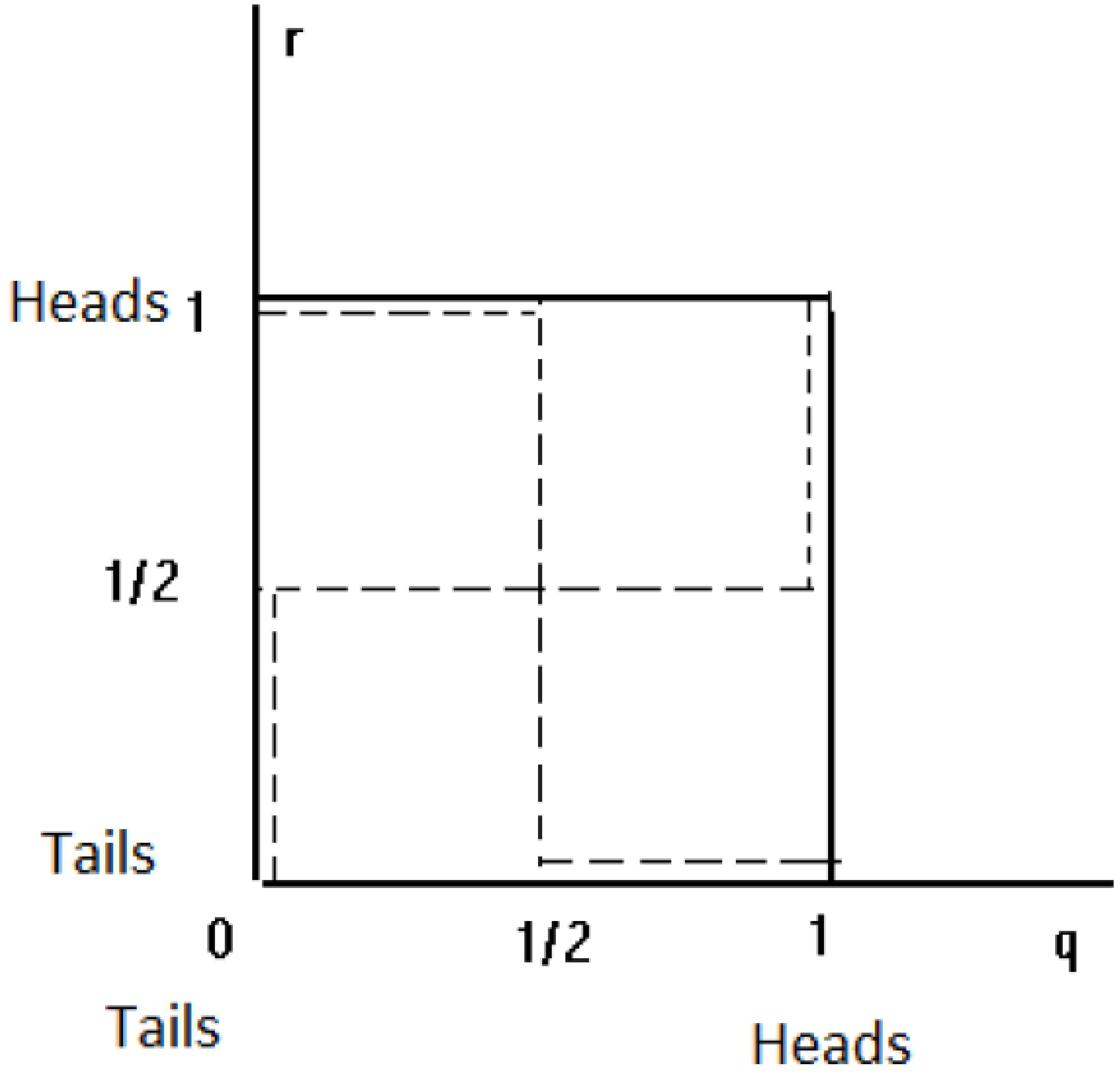
Vice versa, the best reply correspondence of Child 2 is given by,

$$q = \beta_2(r,q) = \begin{cases} 1 & if\ 2r-1 > 1-2r \Leftrightarrow r > \frac{1}{2} \\ [0,1] & if\ 2r-1 = 1-2r \Leftrightarrow r = \frac{1}{2} \\ 0 & if\ 2r-1 < 1-2r \Leftrightarrow r < \frac{1}{2} \end{cases}$$

Nash equilibrium in Matching Pennies

The two best reply correspondences are plotted in the graph. The fixed point of $\beta(r,q) \equiv \beta_1(r,q) \times \beta_2(r,q)$ is located in the intersection.

The unique equilibrium is in (completely) mixed strategies $\left(\frac{1}{2}, \frac{1}{2}\right)$ for both players.



The Battle of the Sexes

In the *Battle of Sexes* game, imagine that Kim (player 1) and Jordan (player 2) make simultaneous decisions about where to go out. Each one can select one of two options: attend the "Opera" or a "Football" match. They hope to meet, and receive zero utility from attending separate events. However, Kim prefers to see the match, while Jordan favors the Opera. The payoff matrix is given by:

		$\boldsymbol{player\ 2\ (Jordan)}$	
		$Opera$	$Football$
$\boldsymbol{player\ 1\ (Kim)}$	$Opera$	$1,2$	$0,0$
	$Football$	$0,0$	$2,1$

Expected payoffs

Let us assume that,

- Kim assigns probability r to watching Football.
- Jordan assigns probability q to watching Football.

Hence, the expected payoffs of the pure strategies of Kim are,

$$E(\pi_1(Football)) = 2q + (1-q)0 = 2q$$
$$E(\pi_1(Opera)) = 0q + (1-q)1 = 1-q$$

The expected payoffs of the pure strategies of Jordan are in turn,

$$E(\pi_2(Football)) = 1r + (1-r)0 = r$$
$$E(\pi_2(Opera)) = 0r + (1-r)2 = 2(1-r)$$

Best reply correspondences

Consequently, we can write the best reply correspondences in the following way,

$$r = \beta_1(r,q) = \begin{cases} 1 & if\ 2q > 1 - q \Leftrightarrow q > \frac{1}{3} \\ [0,1] & if\ 2q = 1 - q \Leftrightarrow q = \frac{1}{3} \\ 0 & if\ 2q < 1 - q \Leftrightarrow q < \frac{1}{3} \end{cases}$$

$$q = \beta_2(r,q) = \begin{cases} 1 & if\ r > 2(1 - r) \Leftrightarrow r > \frac{2}{3} \\ [0,1] & if\ r = 2(1 - r) \Leftrightarrow r = \frac{2}{3} \\ 0 & if\ r < 2(1 - r) \Leftrightarrow r < \frac{2}{3} \end{cases}$$

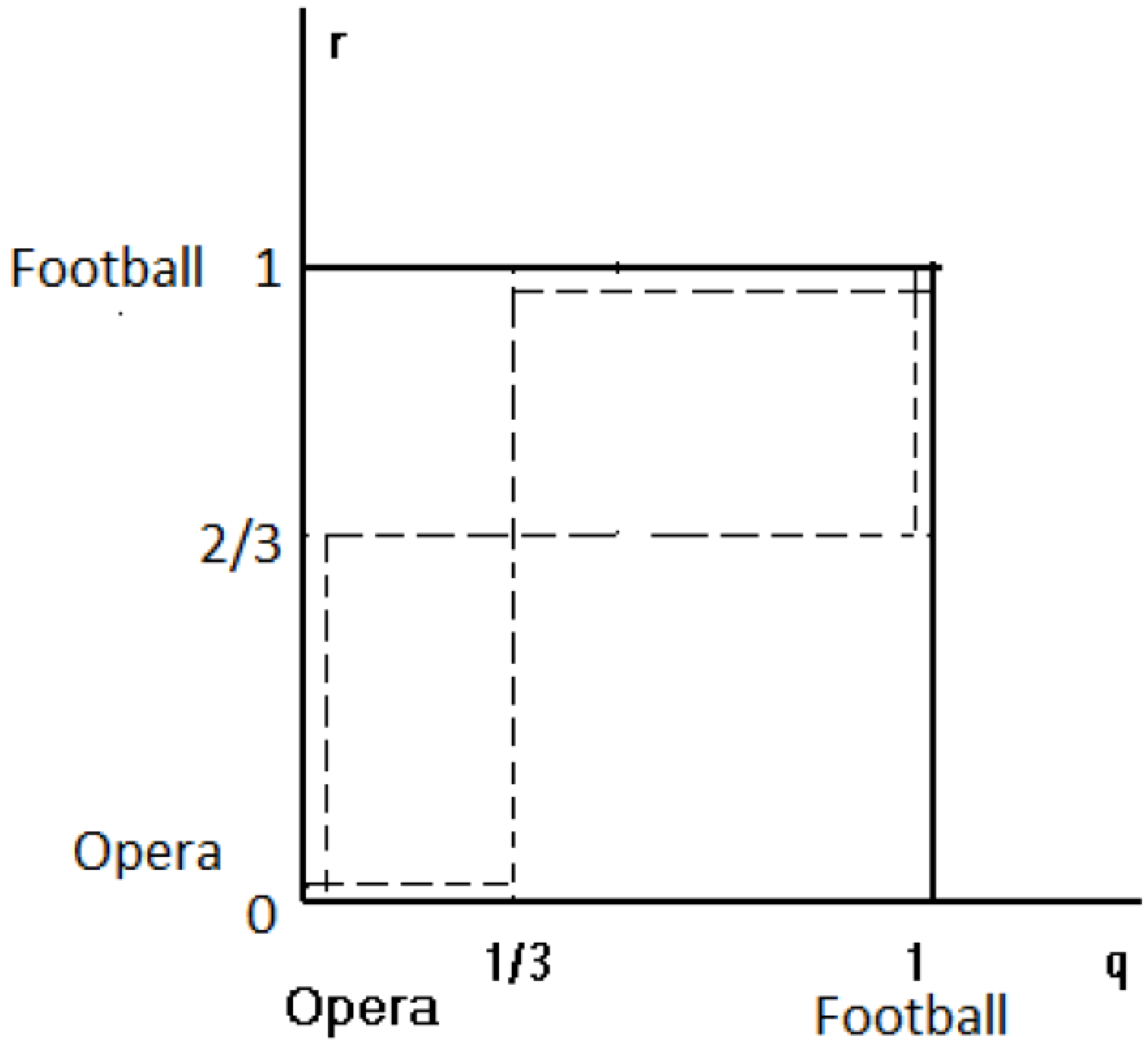
Nash equilibrium in Battle of the Sexes

It is clear that the Nash equilibria, given by the three fixed points of $\beta(r,q) \equiv \beta_1(r,q) \times \beta_2(r,q)$ are:

Pure strategy equilibrium $(\text{Football}, \text{Football}) \equiv (1,1)$

Pure strategy equilibrium $(\text{Opera}, \text{Opera}) \equiv (0,0)$

Completely mixed strategy equilibrium $\left(\frac{1}{3}, \frac{2}{3}\right)$



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